



NEW YORK UNIVERSITY

Institute of Mathematical Sciences

Division of Electromagnetic Research

RESEARCH REPORT No. BR-37

The Exponential Solution for the Homogeneous Linear Differential Equation of the Second Order

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Mathematics Division

Air Force Office of Scientific Research

Contract No. AF 49(638)-229

Project No. 47500

FEBRUARY, 1961

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ABSTRACT

A linear second order differential equation may be considered as a 2×2 system of first order equations. The question is whether the solutions of this system can be written in the form $\exp \Omega$ where Ω is a 2×2 matrix. A motivation for the problem is given, based on the question of defining "lump constants" for an inhomogeneous layer. Conditions necessary for the existence of Ω are given for a variety of circumstances.

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1. Introduction and Motivation

We shall consider the propagation of a plane wave which passes from a homogeneous medium through an inhomogeneous layer into another homogeneous medium. In simple cases (normally incident acoustical or properly polarized electromagnetic waves) the mathematical description of such a wave can be given by solving a differential equation

$$(1.1) \quad \frac{d^2 y}{dx^2} + Q(x)y = 0$$

where

$$Q(x) = k_0 \text{ for } x < 0$$

$$Q(x) = k_1 \text{ for } x > x_0$$

$$Q(x) \text{ variable for } 0 \leq x \leq x_0$$

and where k_0, k_1 are constants.

If we are merely interested in the behavior of the wave (that is, in $y(x)$) for $x < 0$ and for $x > x_0$ - in other words, if we consider only the transmission and reflection properties of the 'layer' $0 \leq x \leq x_0$, then we do not need a full knowledge of the solutions of (1) for $0 \leq x \leq x_0$. All we need is the following information.

Let $y_1(x), y_2(x)$ be the solutions of (1) which are determined respectively by the initial conditions

$$\begin{aligned} y_1(0) &= 1 & y_2(0) &= 0 \\ y_1'(0) &= 0 & y_2'(0) &= 1. \end{aligned}$$

Find the matrix

$$(1.2) \quad Y(x_0) = \begin{pmatrix} y_1(x_0) , y_2(x_0) \\ y'_0(x_0) , y'_2(x_0) \end{pmatrix} .$$

(Here a prime denotes the derivative with respect to x .) Clearly, if $Y(x_0)$ is known, we can compute $y(x)$ for all x outside of the interval $0 \leq x \leq x_0$ under a wide variety of conditions. A particularly important case is the following one which corresponds to an incoming wave for which the reflection and the transmission coefficients are to be computed under the assumption that both the function defining the wave and its derivative are continuous everywhere. We mean the case where $y(x)$ is a solution of (1) such that

$$y(x) = \exp(i \sqrt{k_0} x) + \rho \exp(-i \sqrt{k_0} x) \quad \text{for } x < 0$$

$$y(x) = \tau \exp(i \sqrt{k_1} x) \quad \text{for } x > x_0$$

$$y(x), y'(x) \text{ continuous everywhere.}$$

The quantities ρ and τ will be called, respectively, the reflection and the transmission coefficients. They can be computed from the relations

$$(1.3) \quad \rho [\eta_1 - i \sqrt{k_0} \eta_2] - \tau \epsilon = -i \sqrt{k_0} \eta_2 - \eta_1$$

$$(1.4) \quad \rho [\eta'_1 - i \sqrt{k_0} \eta'_2] - \tau i \sqrt{k_1} \epsilon = -i \sqrt{k_0} \eta'_2 - \eta'_1 ,$$

where

$$\eta_1 = y_1(x_0), \quad \eta'_1 = y'_1(x_0), \text{ etc.}, \quad \epsilon = \exp(i \sqrt{k_1} x_0).$$

In the simplest case where $k_0 = k_1 > 0$, $\eta_1, \eta'_1, \eta_2, \eta'_2$ real, we see that the equations (3), (4) will always have a unique solution, since the determinant

of these equations for ρ , τ , can vanish only if

$$\eta_2' = -\eta, \quad \eta_1' = k_0 \eta_2,$$

in which case

$$\eta_1 \eta_2' - \eta_2 \eta_1' = -\eta_1^2 - k_0 \eta_2^2 < 0$$

which would contradict the relation

$$(1.5) \quad \eta_1 \eta_2' - \eta_2 \eta_1' = 1.$$

Since the transmitting and reflecting properties of our 'layer' depend on only the four constants $\eta_1, \dots, \eta_2$, we may try to replace our layer (i.e., the function $Q(x)$ for $0 \leq x \leq x_0$) by an equivalent constant structure. For this purpose, we cannot replace $Q(x)$ by a constant in the interval $(0, x_0)$, since this may be ⁱⁿcompatible with the specific character of the matrix $Y(x_0)$. However, if we write (1) in the form

$$\begin{aligned} \frac{dy_1}{dx} &= z_1, & \frac{dy_2}{dx} &= z_2 \\ \frac{dz_1}{dx} &= -Q y_1, & \frac{dz_2}{dx} &= -Q y_2, \end{aligned}$$

we obtain the matrix equation

$$(1.6) \quad \frac{dY}{dx} = AY,$$

where

$$(1.7) \quad Y = \begin{pmatrix} y_1 & y_2 \\ z_1 & z_2 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ -Q & 0 \end{pmatrix}$$

Now we can try to replace the matrix A by a constant matrix $C(x_0)$ which, naturally, depends on x_0 . Obviously, we must choose C such that

$$(1.8) \quad \exp[x_0 C(x_0)] = Y(x_0)$$

since the equation

$$(1.9) \quad \frac{d\tilde{Y}}{dx} = C \tilde{Y}$$

has the solution

$$(1.10) \quad \tilde{Y} = \exp xC$$

if we postulate that $Y = I = \text{identity}$ for $x = 0$.

As long as we keep x_0 constant, no real problem arises since it is always possible to find a matrix C which satisfies (8) for a given $Y(x_0)$. This is due to the fact that the determinant of $Y(x_0)$ equals 1 and that any non-singular matrix has (infinitely many complex) logarithms. However, if we wish to satisfy (8) for a variable x_0 , and if we try to do this in such a manner that $C(x_0)$ depends continuously on x_0 , we arrive at the following questions.

(1) Let $Q(x)$ be a real valued continuous function of x which is defined for $-\infty < x < +\infty$. Does there exist a matrix

$$(1.11) \quad \Omega(x) = \begin{pmatrix} \omega & \varphi \\ \psi & -\omega \end{pmatrix}, \quad \Omega(0) = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where φ, ψ, ω are continuous for all values of x , such that the system defined by (6) and (7) has the solution (for all real x)

$$(1.12) \quad Y = \exp \Omega.$$

(ii) Let $Q(x)$ be an entire analytic function of the complex variable x . Is it possible to find entire analytic functions φ, ψ, ω of x such that the matrix $\Omega(x)$ defined by (11) has the property that $Y = \exp \Omega$ solves (6), (7) ?

Clearly, Problem (i) arises immediately from the preceding discussion of the theory of inhomogeneous layers. We have added problem (ii) because, from a purely mathematical point of view, it appears to be a natural analogue of Problem (i).

We shall give partial answers to both problems. It is to be expected that the answer to Problem (ii) is a 'No', unless Q is a constant. All we can show here is that, if Q is a polynomial and Ω is entire, then Q is constant (Theorem 3).

2. Relations between A , Y , and Ω .

In this section, we shall give a survey of relations between A , Y , Ω and their elements. Some of these relations will be needed later. The proofs will be merely indicated.

Let Y , A , Ω be defined respectively by (1.7), (1.11), and (1.12). Using (1.12) and

$$(2.1) \quad \Omega^2 = \Delta^2 I,$$

where I is the unit matrix and where

$$(2.2) \quad \Delta^2 = \omega^2 + \varphi\psi$$

we find that

$$(2.3) \quad Y = \frac{\sinh \Delta}{\Delta} \Omega + \cosh \Delta I.$$

As a consequence of (2.3), we have the relations

$$(2.4) \quad y_1 + z_2 = 2 \cosh \Delta$$

$$(2.5) \quad \omega = \frac{\Delta}{2 \sinh \Delta} (y_1 - z_2), \quad \varphi = \frac{\Delta y_2}{\sinh \Delta}, \quad \psi = \frac{\Delta z_1}{\sinh \Delta}$$

Ω satisfies a non-linear differential equation of the first order which may be written as follows (see Magnus^[1]): Let $P(z)$ be the power series defined by

$$P(z) = \sum_{n=0}^{\infty} p_n z^n = \frac{z}{1 - e^{-z}}$$

and let A, Ω^n be defined recursively by

$$A, \Omega^0 = A, \quad |A, \Omega| = A\Omega - \Omega A,$$

$$A, \Omega^n = A, \Omega^{n-1} \Omega - \Omega A, \Omega^{n-1} \quad \text{for } n = 2, 3, \dots$$

Then

$$(2.6) \quad \frac{d\Omega}{dx} = \sum_{n=0}^{\infty} p_n A, \Omega^n.$$

Using (2.1), we can derive from (2.6) the formulas

$$(2.7) \quad \frac{d\Omega}{dx} = A + (A\Omega - \Omega A) + \left(\frac{\Delta}{e^{2\Delta} - 1} - 1 + \Delta \right) (\Delta^2 A - \Omega A \Omega) / (2\Delta^2)$$

$$(2.8) \quad \Omega \frac{d\Omega}{dx} - \frac{d\Omega}{dx} \Omega = (-\Omega A + A\Omega)(\Omega + \Delta \coth \Delta I)$$

$$(2.9) \quad \Omega \frac{d\Omega}{dx} + \frac{d\Omega}{dx} \Omega = 2 \Delta \frac{d\Delta}{dx} I = \Omega A + A\Omega.$$

By taking the traces on both sides of (2.9), we obtain

$$(2.10) \quad 2 \Delta \frac{d\Delta}{dx} = \psi - Q\varphi.$$

All of the elements of Ω (and therefore all those of Y) can be expressed in terms of Q and Δ alone, and Δ itself satisfies a differential equation with coefficients depending only on Q and its derivatives. The formulas in question are:

$$(2.11) \quad \varphi = - \frac{2\Delta}{Q'} \left[\Delta'' + \coth \Delta (\Delta'^2 + Q) \right]$$

$$(2.12) \quad \psi = (\Delta^2)' - \frac{2Q\Delta}{Q'} \left[\Delta'' + \coth \Delta (\Delta'^2 + Q) \right]$$

$$(2.13) \quad \omega = \Delta \coth \Delta + \frac{\Delta}{\sinh \Delta} \left\{ \frac{2}{Q'} \left[(\cosh \Delta)'' + Q \cosh \Delta \right] \right\}'.$$

Here a prime (') denotes the derivative with respect to x . Clearly, the last three formulas lead to a differential equation connecting Q and Δ since $\Delta^2 = \omega^2 + \varphi\psi$. However, such a differential equation can be obtained more easily by observing that

$$(2.14) \quad \left[y_1'' + z_2'' + Q(y_1 + z_2) \right] / Q'$$

satisfies the same differential equation as y_1 and y_2 .

This leads to

$$(2.15) \quad \left[\frac{(\cosh \Delta)'' + Q \cosh \Delta}{Q'} \right]'' = - Q \frac{(\cosh \Delta)'' + Q \cosh \Delta}{Q'},$$

a differential equation of the fourth order for Δ with coefficients depending on Q (provided that Q is three times differentiable).

We can prove (2.11) to (2.13) in the following manner. By taking the traces on both sides of (2.3), differentiating repeatedly and using (1.6), we find

$$y_1 + z_2 = 2 \cosh \Delta$$

$$(1) \quad y_1' + z_2' = z_1 - Qy_2 = 2 \Delta' \sinh \Delta$$

$$(2) \quad y_1'' + z_2'' = -Q(y_1 + z_2) - Q'y_2 = -2Q \cosh \Delta - Q'y_2$$

and these equations show that

$$(3) \quad y_2 = -\frac{2}{Q'} \left[\Delta'' \sinh \Delta + (Q + \Delta'^2) \cosh \Delta \right]$$

$$(4) \quad z_1 = Qy_2 + 2\Delta' \sinh \Delta$$

$$(5) \quad y_1 = 2 \cosh \Delta - y_2' \quad , \quad y_2' = z_2 \quad .$$

From here, it is easy to derive (2.11) to (2.13) by using (2.3).

For later purposes, we also note here the formulas arising from (2.3) by differentiation. They are

$$(2.16) \quad y_1' = z_1 = \frac{\sinh \Delta}{\Delta} \psi = \left[\Delta' + (\omega/\Delta)' \right] \sinh \Delta + \frac{\omega \Delta'}{\Delta} \cosh \Delta$$

$$(2.17) \quad y_2' = z_2 = -\frac{\sinh \Delta}{\Delta} \omega + \cosh \Delta = \left(\frac{\sinh \Delta}{\Delta} \right)' \varphi + \frac{\sinh \Delta}{\Delta} \varphi'$$

$$(2.18) \quad \begin{aligned} z_1' &= -Qy_1 = -Q \left(\omega \frac{\sinh \Delta}{\Delta} + \cosh \Delta \right) \\ &= \psi' \frac{\sinh \Delta}{\Delta} + \psi \left(\frac{\sinh \Delta}{\Delta} \right)' . \end{aligned}$$

3. The Case $Q(x) < 0$, for Real x .

We shall prove the following.

Theorem 1. Let $Q(x)$ be real and negative for all real values of x . If $Q(x)$ is continuous, then there exists a matrix Ω with elements which depend continuously on x such that, for all real x ,

$$Y = \exp \Omega, \quad \Omega(0) = 0.$$

Pr of: Using (2.5), we merely have to show that Δ never assumes any one of the values

$$(3.1) \quad \Delta_n = n\pi i, \quad (n = \pm 1, \pm 2, \dots, n \neq 0).$$

This will be true (because of (2.4)) if $y_1 + z_2 > 2$, for all $x \neq 0$. The initial conditions for y_1 and $y_1' = z_1$, together with the differential equation, show that y_1 has a minimal value at $x = 0$. The same is true for z_2 , since $z_2 = 1$ at $x = 0$ proves that $y_2 > 0$ for $x > 0$ and $y_2 < 0$ for $x < 0$ if $|x|$ is sufficiently small. Therefore

$$z_2' = -Qy_1$$

is indeed positive for small positive x , negative for small negative x and vanishes for $x = 0$.

This shows that, in the neighborhood of $x = 0$, $y_1 + z_2 > 2$ for $x \neq 0$. But it is even true that $y_1 > 1$, $z_2 > 1$ for all $x \neq 0$. In fact, let x_1 be the smallest positive value of x for which $y_1 - 1 = 0$. Then there exists a positive $x_0 < x_1$ such that $y_1'(x_0) = 0$. However, for $0 < x \leq x_0$, y_1 is not decreasing and therefore $y_1 > 1$. Therefore $y_1' = z_1$, is actually increasing in this interval since $z_1' = y_1'' = -Qy_1 > -Q > 0$. This excludes

any zeros of y_1' for $x > 0$. By using similar arguments, it can be shown that $y_1 > 1$ for $x < 0$ and that $z_2 > 1$ for all $x \neq 0$. This proves Theorem 1 completely.

4. The Case $Q(x) > 0$ for Real x .

In this section, we shall make the following assumptions:

1. $Q(x)$ is a positive differentiable function for all values of x .
2. The differential equation (1.6) has a solution

$$(4.1) \quad Y = \exp \Omega$$

where the elements of Ω are differentiable (and, necessarily, real) functions for x for all x , such that Ω vanishes for $x = 0$.

3. Let y_1, y_2 be the normalized solutions of (1.1) defined by the initial conditions

We shall assume that

$$(4.2) \quad \int_{-\infty}^0 \frac{dx}{y_1^2 + y_2^2} > \pi, \quad \int_0^{\infty} \frac{dx}{y_1^2 + y_2^2} > \pi.$$

Also, we shall use the following notations. We shall denote 0 by x_0 and we shall define x_n by

$$(4.3) \quad \int_0^{x_n} \frac{dx}{y_1^2 + y_2^2} = n\pi, \quad n = \pm 1, \pm 2, \dots$$

Of course, x_n may not exist for $|n| > 1$.

The notations of Sections 1 and 2 will be used without further comment.

As an abbreviation, we define θ by

$$(4.4) \quad 2 \cosh \Delta = y_1 + y_2' = \theta.$$

Now we can state

Theorem 2. If the assumptions 1, 2, and 3 are satisfied, then

$$(4.5) \quad Y = (-1)^n I \quad \text{for } x = x_n$$

$$(4.6) \quad \theta^2 < 4 \quad \text{for } x \neq x_n$$

and there exists no value of x between x_{n-1} and x_n for which

$$Y = \pm I.$$

To prove Theorem 2, we need the following Lemmas.

Lemma 1. In every interval on the x -axis in which $\theta = 2 \cosh \Delta$ satisfies the inequality

$$\theta^2 - 4 < 0,$$

θ is a monotonic function of x such that $\theta' = d\theta/dx$ is different from zero and has the same sign in the whole interval.

Proof: If $\theta' = 0$, $y_1' + y_2'' = y_1' - Qy_2 = 0$. Therefore, y_1' and y_2 have the same sign (since $Q > 0$). Therefore, if $\theta' = 0$,

$$(4.7) \quad \theta^2 - 4 = (y_1 - y_2')^2 + 4y_1y_2' - 4 = (y_1 - y_2')^2 + 4y_1'y_2 > 0$$

unless

$$y_1 = y_2', \quad y_1' = y_2 = 0.$$

(Note that $y_1'y_2 = 0$ and $y_1' - Qy_2 = 0$ implies $y_1' = y_2 = 0$).

Lemma 2. Let $Q > 0$ for all x and assume that Ω is differentiable everywhere. If, for a value $x_0 > 0$, $\theta^2 < 4$, $\theta' < 0$ and the real quantity ~~is~~ $D = i\Delta$ is, in absolute value, less than π , then $D' \neq 0$ for all $x > x_0$. Similarly, $D' \neq 0$ for all $x \leq x'_0 < 0$. If, for $x = x'_0$, we have $\theta^2 < 4$, $\theta' > 0$, $|D| < \pi$.

Proof: Whenever $\theta^2 < 4$, $\Delta^2 < 0$. Since we know that for $x = x_0$ this condition is satisfied, we have

$$(4.8) \quad \theta = 2 \cos D, \quad 0 < D < \pi.$$

Since

$$(4.9) \quad \theta' = -2D' \sin D < 0$$

it follows that

$$D' > 0 \text{ if } D > 0, \quad D' < 0 \text{ if } D < 0.$$

Therefore, D increases in absolute value. Therefore, $D \neq 0$ if, for the first time, $\theta^2 = 4$ for $x > x_0$. We have to show that $D' \neq 0$ even at this point. We see from (2.8) that, for $\sinh \Delta = 0$,

$$(4.10) \quad A\Omega - \Omega A = 0.$$

This implies that, at the point under consideration,

$$(4.11) \quad \psi + Q\varphi = 0, \quad \omega = 0.$$

On the other hand, we see from (2.10) that

$$(4.12) \quad 2 \Delta \Delta' = -2 D D' = \psi - Q\varphi.$$

Now $D' = 0$ would, together with $\psi + Q\varphi = 0$, imply that $\varphi = \psi = 0$. But $\omega = 0$ and

$$D^2 = -\varphi\psi \neq 0.$$

Therefore, $D' \neq 0$. From this point on we can use the same argument again, since $|D|$ is increasing, θ' will be positive after we have passed the point where $\theta^2 = 4$, $|D| = \pi$, until we reach a point where $|D| = 2\pi$. Again we can show that there $D' \neq 0$ and so we see that Lemma 2 is true. Incidentally, we have proved

Lemma 3. If the conditions of Lemma 2 are satisfied, then,

for all $x > x_0$ (and for all $x < x'_0$), $\theta^2 \leq 4$ and $\Delta^2 < 0$.

Now we can prove Theorem 2 as follows. According to Lemma 3, $\theta^2 \leq 4$ for all x if we can show that $\theta^2 < 4$ in a sufficiently small neighborhood of $x = 0$. At $x = 0$ we have

$$(4.13) \quad \begin{cases} \theta = y_1 + y_2' = 2 \\ \theta' = y_1' - Qy_2 = 0 \\ \theta'' = y_1'' - Qy_2' - Q'y_2 = -Q(y_1 + y_2) - Q'y_2 = -2Q. \end{cases}$$

Since $Q > 0$, we see that $\theta' < 0$ in the neighborhood of $x = 0$, $x > 0$ and $\theta' > 0$ for $x < 0$ and $|x|$ sufficiently small. Therefore, we have $\theta^2 \leq 4$ everywhere, and we see that either $|\Delta^2| < \pi^2$ for all x or that there exist certain values of n and of $x = x'_n$ such that $\Delta = i\pi n$ for $x = x'_n$. We have to show that these values of x'_n are exactly the values x_n defined in (4.3) and that, for $\Delta = n\pi i$, (4.5) must necessarily follow from (2.3).

To show that $x_n = x'_n$, we define R and ρ by

$$(4.14) \quad \rho^2 = y_1^2 + y_2^2, \quad \rho > 0$$

$$(4.15) \quad R = \int_0^x \frac{dt}{\rho^2(t)}$$

Then we have

$$(4.16) \quad y_1 = \rho \cos R, \quad y_2 = \rho \sin R$$

$$(4.17) \quad \theta = (\rho + \rho^{-1}) \cos R + \rho' \sin R.$$

We shall now prove that $x'_1 = x_1$. The proof for the general case follows exactly the same pattern. First, we observe that $\theta = 2$ for $x = 0$ and $\theta \leq -2$ for $x = x_1$. Since $\theta^2 \leq 4$ for all x , we must have $\theta = -2$ for $x = x_1$ which implies $\rho(x_1) = 1$. Since, for $\theta = -2$, $Y = -I$, we must have $y'_1 = 0$ for $x = x_1$. Therefore, $\rho'(x_1) = 0$. Now x_1 must be the smallest positive value of x for which $\theta = -2$. Since we know that $\theta^2 < 4$, $\theta' < 0$ for sufficiently small positive x , θ' cannot vanish according to Lemma 1 until $\theta = -2$, and at this point $x = x'_1$ according to (4.16). We must have $y_2 = 0$, according to (2.3). Therefore, according to (4.16), we must have $R = \pi$ at $x = x'_1$, but by definition of x_1 , $R = \pi$ for $x = x_1$ which proves $x_1 = x'_1$ and Theorem 2.

5. The Analytic Case.

We shall prove

Theorem 3. If $Q(x)$ is a polynomial and if the elements

ϕ, ψ, ω of Ω are entire analytic functions of x , then Q must be a constant.

The proof will be given by showing that the derivative Q' of Q will vanish at the 'critical points' where

$$(5.1) \quad \Delta = n\pi i, \quad (n = \pm 1, \pm 2, \pm 3, \dots; n \neq 0).$$

This is the statement of Formula (5.7) in Lemma 4. It proves Theorem 3 if we can show that Δ will assume a critical value for infinitely many values of x . In turn, this follows from Picard's Theorem, unless Δ is a constant. In fact, the entire function

$$\Delta^2 = \omega^2 + \varphi\psi$$

can leave out only finitely many of the values of $-n^2\pi^2$ unless Δ is a constant. If Δ is a constant, then $\Delta = 0$, since, for $x = 0$,

$$Y = \frac{\sinh \Delta}{\Delta} \Omega + \cosh \Delta I = I,$$

which implies that either $\sinh \Delta = 0$, $\Delta \neq 0$ or $\varphi = \psi = 0$, $\Delta = \pm \omega$, and

$$\cosh \Delta \pm \sin \Delta = I,$$

which again implies $\sinh \Delta = 0$. Since Δ is supposed to be different from the values (5.1), we have $\Delta = 0$. But if Δ vanishes identically, we see from (2.11) to (2.13) that

$$(5.2) \quad \varphi = -2Q/Q', \quad \psi = -2Q^2/Q'$$

$$(5.3) \quad \omega = 1 + 2(Q/Q')',$$

and therefore

$$(5.4) \quad \begin{aligned} \Delta^2 &= \varphi\psi + \omega^2 \\ &= 4Q^3(Q')^{-2} + \left[1 + 2(Q/Q')'\right]^2 = 0. \end{aligned}$$

It can be shown that (5.4) is not only a necessary but also a sufficient condition for Q to guarantee that $\Delta \equiv 0$. However, all we need here is the remark that an entire function Q satisfying (5.4) cannot vanish because the assumption that

$$Q = (x - x_0)^n + \text{higher terms}$$

leads to a contradiction with (5.4), and that therefore (5.2) is incompatible with (5.4) since $\phi = 0$ for $x = 0$ by assumption. Therefore, Δ cannot vanish identically and this, in turn, shows that Δ will assume infinitely many critical values (5.1).

As announced above, Theorem 3 follows now from

Lemma 4. At a critical point (5.1), the following relations hold

$$(5.5) \quad \Delta'^2 + Q = 0$$

$$(5.6) \quad \Omega = \frac{\Delta}{\Delta'} A,$$

$$(5.7) \quad Q' = 0.$$

Proof. We obtain (5.5) immediately from (2.11) since $\sinh \Delta = 0$, $\Delta \neq 0$, and Q finite at the critical point. To obtain (5.6), we may use (2.17) to show that $\phi = \Delta/\Delta'$ and (2.16) to prove that $\omega = 0$ since $\Delta' \neq 0$ if ϕ is finite. Finally, (2.18) yields $\psi = -Q\Delta/\Delta'$ once we know that $\omega = 0$.

Equation (5.7) is more difficult to prove. We shall derive (5.7) from a passage to the limit

$$\Delta \rightarrow \Delta_n = n\pi i$$

in (2.8). If we denote the value of a quantity at $\Delta = \Delta_n$ by a subscript n , we find that (2.8) gives us the relation

$$\begin{aligned}
 (5.8) \quad \Omega_n \Omega'_n - \Omega'_n \Omega_n &= \left(\lim_{\Delta \rightarrow \Delta_n} \frac{A\Omega - \Omega A}{\sinh \Delta} \right) \Delta_n \cosh \Delta_n \\
 &= \frac{\Delta_n}{\Delta'_n} \left(A_n \Omega'_n - \Omega'_n A_n \right) + \frac{\Delta_n}{\Delta'_n} \left(A'_n \Omega_n - \Omega_n A'_n \right)
 \end{aligned}$$

since $\Omega_n = (\Delta_n/\Delta'_n)A_n$ according to the first equation (5.6). The same relation, together with (5.8) shows that

$$(5.9) \quad A'_n \Omega_n - \Omega_n A'_n = 0$$

and therefore

$$(5.10) \quad A'_n A_n = A_n A'_n$$

which immediately gives us (5.7). We could also prove $\Omega'_n = A_n$ by letting $\Delta \rightarrow \Delta_n$ in (5.7) , using ~~the first equation~~ ^{5.7} (5.6) as well as (5.7). But we do not need this.

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